Introduction to Quantum Mechanics

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4^{th} High Performance and Disruptive Computing in Remote Sensing School

JAGIELLONIAN UNIVERSITY IN KRAKÓW

Quantum Cosmos Lab

Course material

09:30 - 11:00	**Introduction to quantum mechanics**
11:00 - 11:30	Break
11:30 - 13:00	Introduction to quantum computation
13:00 - 14:30	Lunch
14:30 - 15:30	Quantum machine learning
15:30 - 16:00	Break
16:00 - 17:00	Quantum algorithms for Remote Sensing

Introduction to quantum mechanics

- Review on complex numbers
- Hilbert space
- Postulates of quantum mechanics
- Unitary evolution
- Simple Quantum Gates

Quiz

Join at menti.com | use code **6454 3204**

Complex numbers - basics

Basic definition

$$z = \underbrace{a}_{\text{real part}} + i \underbrace{b}_{\text{imaginary part}}$$

$$Re[z] = a,\ \ Im[z] = b$$

$$i^2 = -1 \rightarrow \sqrt{-1} = i$$

Complex conjugation

$$z^* = a - ib$$

$$Re[z] = \frac{1}{2}(z + z^*)$$

$$Im[z] = \frac{1}{2}(z - z^*)$$

Basic operations

Addition

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

Multiplication

$$z_1 \cdot z_2 = (a_1 + ib_1) \cdot (a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Modulus

$$|z| = \sqrt{z \cdot z^*} = \sqrt{a^2 + b^2}$$

Example

$$z = 2 - 3i$$

$$Re[z] = 2,\ \ Im[z] = -3$$

$$z^* = 2 + 3i, |z| = \sqrt{4 + 9} = \sqrt{13}$$

Complex numbers - polar form

Basic definition

$$z = \underbrace{r}_{modulus} \cdot e^{i \overbrace{\varphi}^{argument}}$$

$$Re[z] = r \cdot cos\varphi, \;\; Im[z] = r \cdot sin\varphi$$

$$z^* = r \cdot e^{-i\varphi}$$

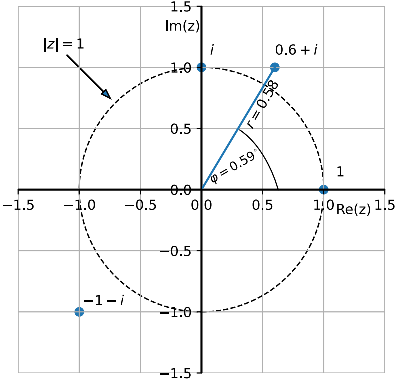
Basic operations

Multiplication

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$$

Modulus

$$|z| = r$$



Complex matrices

Basic definitions

$$a, b, c, d \in \mathbb{C}, M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$M^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, M^* = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix},$$

$$M^\dagger = (M^T)^* = (M^*)^T = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix},$$

$$(M_1 \cdot M_2)_{ij} = \sum_k (M_1)_{ik} \cdot (M_1)_{kj}$$

In general

$$M_1 \cdot M_2 \neq M_2 \cdot M_1$$

$$[M_1, M_2] = M_1 \cdot M_2 - M_2 \cdot M_1$$

Hermitian matrix

- $H = H^\dagger$
- Generalization of symmetric matrices
- All eigenvalues real

Unitary matrix

- $U \cdot U^\dagger = \mathbb{I}$
- $U^{-1} = U^\dagger$
- Generalization of orthogonal matrices
- $|det(U)| = 1, \; det(U) = e^{i\varphi}$

Postulate I

In quantum mechanics the state of a physical system is represented by a vector in a Hilbert space $\mathcal{H}$: a complex vector space with an inner product.

(This course - only finite dimensional Hilbert spaces)

Dirac notation

Vector: 'ket' $|\psi\rangle$ Dual vector: 'bra' $\langle\varphi|$
Customarily, for $d = dim(\mathcal{H})$:

$$|0\rangle = \left.\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right\} d,\ |1\rangle = \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \dots, |d-1\rangle = \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}$$

$\langle 0| = [1\ 0\ \dots\ 0]$, $\langle\psi| = (|\psi\rangle)^\dagger$

Hilbert space

A Hilbert space $\mathcal{H}$ is a complex inner product space.

- Linear combination (superposition):

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle, \ |\varphi\rangle = \sum_{i=0}^{d-1} \beta_i |i\rangle,$$

- Inner product (bra-ket):

$$\langle i|j\rangle = \delta_{ij}$$

$$\langle\psi|\varphi\rangle = \sum_{ij} \alpha_i^* \beta_j \langle i|j\rangle = \sum_i \alpha_i^* \beta_i$$

- Assumption of normalization:

$$||\psi||^2 = \langle\psi|\psi\rangle = 1$$

State examples

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{d-1} \end{bmatrix}$$

$$|\varphi_1\rangle = \frac{1}{\sqrt{30}} (|0\rangle + 2i|1\rangle + 3|2\rangle + 4i|3\rangle)$$

$$|\varphi_2\rangle = \frac{1}{2} ((1+i)|0\rangle + i|1\rangle + |2\rangle)$$

$$|\varphi_1\rangle = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2i \\ 3 \\ 4i \end{bmatrix}, \ |\varphi_2\rangle = \frac{1}{2} \begin{bmatrix} 1+i \\ i \\ 1 \\ 0 \end{bmatrix}$$

$$\langle\varphi_1|\varphi_2\rangle = \frac{1}{2\sqrt{30}} ((1+i) + 2 + 3 + 0) = \frac{6+i}{2\sqrt{30}}$$

‘Matrix’ multiplication

$$\langle\varphi_1|\varphi_2\rangle = (|\varphi_1\rangle)^\dagger|\varphi_2\rangle =$$

$$= \frac{1}{\sqrt{30}}[1, -2i, 3, -4i] \cdot \frac{1}{2}\begin{bmatrix}1+i\\ i\\ 1\\ 0\end{bmatrix} = \frac{6+i}{2\sqrt{30}}$$

Exercise 1

1. Find N, such that the state

$$|\varphi\rangle = \frac{1}{N}(|0\rangle + 5i|1\rangle)$$

is normalized, $||\varphi||^2 = \langle\varphi|\varphi\rangle = 1$.

2. Is N unique?

Exercise 1 part 1

1. Find N, such that the state

$$|\varphi\rangle = \frac{1}{N}\left(|0\rangle + 5i|1\rangle\right)$$

is normalized, $||\varphi||^2 = \langle\varphi|\varphi\rangle = 1$.

Answers: $a)\sqrt{26}, \quad b)17i, \quad c)6, \quad d)36$

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Answers: $a)\sqrt{26}$, $b)17i$, $c)6$, $d)36$

$$\langle\varphi|\varphi\rangle = \frac{1}{N^*}(\langle 0| - 5i\langle 1|) \cdot \frac{1}{N}(|0\rangle + 5i|1\rangle) = \frac{26}{|N|^2} = 1 \Rightarrow |N| = \sqrt{26}$$

Exercise 1 part 2

2. Is N unique?
Answers: $a)Yes$, $b)No$

Exercise 1 part 1

1. Find N, such that the state

$$|\varphi\rangle = \frac{1}{N}\left(|0\rangle + 5i|1\rangle\right)$$

is normalized, $||\varphi||^2 = \langle\varphi|\varphi\rangle = 1$.
Answers: $a)\sqrt{26}$, $b)17i$, $c)6$, $d)36$

$$\langle\varphi|\varphi\rangle = \frac{1}{N^*}(\langle 0| - 5i\langle 1|) \cdot \frac{1}{N}(|0\rangle + 5i|1\rangle) = \frac{26}{|N|^2} = 1 \Rightarrow |N| = \sqrt{26}$$

Exercise 1 part 2

2. Is N unique?
Answers: $a)Yes$, $b)No$

No, we only know the modulus of N, but the phase is arbitrary, $N = \sqrt{26}e^{i\alpha}$.
Customarily, we choose N real, $N = \sqrt{26}$.

Qubit

Two state system:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

From normalization condition $|\alpha|^2+|\beta|^2 = 1$
Examples:

$$|0\rangle, \ \alpha = 1, \beta = 0$$

$$|1\rangle, \ \alpha = 0, \beta = 1$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

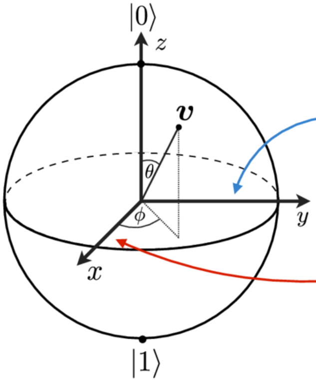
$$|i+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|i-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

Postulate I - Qubit

State on the Bloch sphere

$$|\psi\rangle = cos(\theta/2)|0\rangle + e^{i\varphi} sin(\theta/2)|1\rangle$$



Pole states:

$$|i+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|i-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Tensor product

Hilbert space of a composite system is the tensor product of the Hilbert spaces for the subsystems.

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2,\; dim(\mathcal{H}) = \dim(\mathcal{H}_1) \cdot \dim(\mathcal{H}_2)$$

Customarily:

$$|i\rangle \otimes |j\rangle = |i\rangle|j\rangle = |i\ j\rangle$$

For qubits:

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle, |\varphi\rangle = \beta_0|0\rangle + \beta_1|1\rangle$$

$$|\psi\rangle \otimes |\varphi\rangle = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

$$|\psi\rangle \otimes |\varphi\rangle = \begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix} \otimes \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} = \begin{bmatrix}\alpha_0\beta_0\\ \alpha_0\beta_1\\ \alpha_1\beta_0\\ \alpha_1\beta_1\end{bmatrix}$$

Postulate I - Composite systems

Entangled states

States of the form $|\psi\rangle \otimes |\varphi\rangle$ are product states, all others are entangled states.

Consider:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Interpretation:
A product state $|\psi\rangle \otimes |\varphi\rangle$ has the meaning that system ψ has the property $|\psi\rangle$ and system φ the property $|\varphi\rangle$. For an entangled state one typically cannot assign definite properties to the individual systems ψ and φ.

Exercise 2

Which of the following states is entangled?

$$|\psi\rangle = |00\rangle - |01\rangle - |10\rangle + |11\rangle, \ |\phi\rangle = |00\rangle + |01\rangle - |10\rangle + |11\rangle$$

a) Both are entangled,
b) Only $|\psi\rangle$,
c) Only $|\phi\rangle$,
d) None.

Exercise 2

Which of the following states is entangled?

$$|\psi\rangle = |00\rangle - |01\rangle - |10\rangle + |11\rangle, \ |\phi\rangle = |00\rangle + |01\rangle - |10\rangle + |11\rangle$$

a) Both are entangled,
b) Only $|\psi\rangle$,
c) Only $|\phi\rangle$,
d) None.

Solution

$$|\psi\rangle = |00\rangle - |01\rangle - |10\rangle + |11\rangle,$$

remember the formula $(a-b)^2 = a^2 - ab - ba + b^2$? Mixed terms are negative, hence $|\psi\rangle = (|0\rangle - |1\rangle) \otimes ((|0\rangle - |1\rangle))$ - not entangled.
We would expect that there will be even negative terms if all basis states are present.
$|\phi\rangle = |00\rangle + |01\rangle - |10\rangle + |11\rangle$ - entangled.
Changing the relative phase leads to different states. Only global phase does not matter!

Postulate IIa

Every measurable physical quantity $\mathcal{A}$ is described by a Hermitian matrix A acting in the state space $\mathcal{H}$.
The result of measuring a physical quantity $\mathcal{A}$ must be one of the eigenvalues of the corresponding matrix A.

Postulate IIb

When the physical quantity $\mathcal{A}$ is measured on a system in a normalized state $|\psi\rangle$, the probability of obtaining an eigenvalue (denoted a_n) of the corresponding observable A is given by the amplitude squared of the appropriate state (projection onto corresponding eigenvector).

$$Pr[a_n] = |\langle a_n|\psi\rangle|^2,$$

where $A|a_n\rangle = a_n|a_n\rangle$.

Hermitian matrices as observables

Hermitian matrices have real eigenvalues. Although we work with complex numbers, the measurement outcome is real, like in classical physics.

Pauli matrices and computational basis

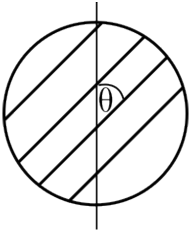
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

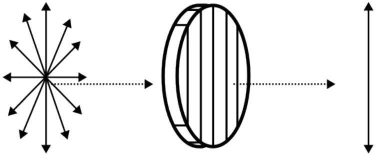
$$Z|0\rangle = 1|0\rangle,\ Z|1\rangle = -1|1\rangle$$

$$X|0\rangle = |1\rangle,\ X|1\rangle = |0\rangle$$

$$X|+\rangle = 1|+\rangle,\ X|-\rangle = -1|-\rangle$$

$$Y|i+\rangle = 1|i+\rangle,\ Y|i-\rangle = -1|i-\rangle$$



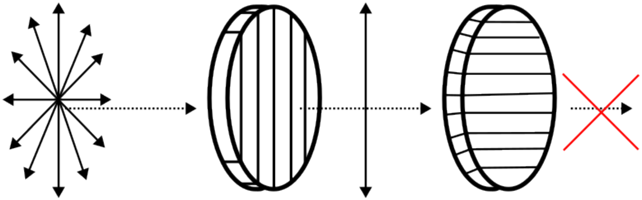


- Two orthogonal polarizations of light, vertical and horizontal, introduce a basis: $|v\rangle, |h\rangle$
- Two dimensional Hilbert space: $\mathcal{H} = span(|v\rangle, |h\rangle)$, $\langle v|h\rangle = 0$
- Any polarization photon state: $|\theta\rangle = cos(\theta)|v\rangle + sin(\theta)|h\rangle$
- The action of the polarizer:
  - ◇ The intensity of light after a polarizer is proportional to the probability that the photon had a correct polarization

$$Pr[v] = |\langle v|\theta\rangle|^2, Pr[h] = |\langle h|\theta\rangle|^2$$

For unpolarized light (Malus' law 1809)

  - ◇ $Pr[v|unpolarized] = \int |\langle v|\theta\rangle|^2 d\theta = \int cos^2(\theta) d\theta = \frac{1}{2}$

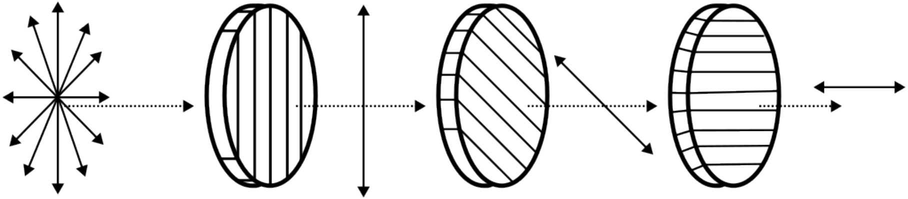


1. Checking the polarization filters: apply two filters orthogonally.
2. After the first polarizer the light is in state $|v\rangle$
3. The second polarizer projects the state on the horizontal orientation eigenvector $|h\rangle$,

$$Pr[h|v] = |\langle h|v\rangle|^2 = 0$$

The light vanishes in this setup.

Postulate I+II - Experiment pt. 3



1. Insert tilted polarizer ($\theta = 45^\circ$),

$$
\begin{array}{rl}
 & |45^\circ\rangle = \frac{1}{\sqrt{2}}|v\rangle + \frac{1}{\sqrt{2}}|h\rangle \\
+ & |-45^\circ\rangle = \frac{1}{\sqrt{2}}|v\rangle - \frac{1}{\sqrt{2}}|h\rangle \\
\hline
 & |v\rangle = \frac{1}{\sqrt{2}}|45^\circ\rangle + \frac{1}{\sqrt{2}}|-45^\circ\rangle
\end{array}
$$

2. After the first polarizer we have state $|v\rangle$, after the second

$$Pr[45^\circ|v] = |\langle 45^\circ|v\rangle|^2 = |\frac{1}{\sqrt{2}}|^2$$

3. After the final polarizer

$$Pr[h|45^\circ] = |\langle h|45^\circ\rangle|^2 = |\frac{1}{\sqrt{2}}|^2$$

4. The intensity in the whole process

$$Pr[v|unpolarized]Pr[45^\circ|v]Pr[h|45^\circ] = \frac{1}{8}$$

Postulate III

The time evolution of the state vector $|\psi(t)\rangle$ is governed by the Schrödinger equation, where $\mathscr{H}$ is the observable associated with the total energy of the system (called the Hamiltonian)

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \mathscr{H}|\psi(t)\rangle$$

Postulate III

The time evolution of a closed system is described by a unitary transformation on the initial state.

$$|\psi(t)\rangle = U(t; t_0)|\psi(t_0)\rangle$$

Exercise 3

Assume that we have a unitary U_{cl} such that it is able to copy an arbitrary one-qubit state to the second register

$$U_{cl}\left(|\psi\rangle \otimes |0\rangle\right) = |\psi\rangle \otimes |\psi\rangle$$

What is the result of the action of U_{cl} on the state $(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle$?
Answers:
a) $\alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$
b) $\alpha|00\rangle + \beta|11\rangle$

Solution?

Both answers seem to be correct...

- First apply U_{cl}, then multiply
  $U_{cl}\left(\left(\alpha|0\rangle+\beta|1\rangle\right)\otimes|0\rangle\right)=\left(\alpha|0\rangle+\beta|1\rangle\right)\otimes\left(\alpha|0\rangle+\beta|1\rangle\right)=\alpha^2|00\rangle+\alpha\beta|01\rangle+\alpha\beta|10\rangle+\beta^2|11\rangle$
- First multiply, then apply U_{cl}
  $U_{cl}\left(\left(\alpha|0\rangle+\beta|1\rangle\right)\otimes|0\rangle\right)=U_{cl}\left(\alpha|00\rangle+\beta|10\rangle\right)=U_{cl}\left(\alpha|00\rangle\right)+U_{cl}\left(\beta|01\rangle\right)=\alpha|00\rangle+\beta|11\rangle$

Solution?

Both answers seem to be correct...

- First apply U_{cl}, then multiply
  $U_{cl}\left((\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle\right) = (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) = \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$
- First multiply, then apply U_{cl}
  $U_{cl}\left((\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle\right) = U_{cl}\left(\alpha|00\rangle + \beta|10\rangle\right) = U_{cl}\left(\alpha|00\rangle\right) + U_{cl}\left(\beta|01\rangle\right) = \alpha|00\rangle + \beta|11\rangle$

No-cloning theorem

There is no unitary operator U on $\mathcal{H} \otimes \mathcal{H}$ such that for all normalized states $|\phi\rangle_A$ and $|e\rangle_B$ in $\mathcal{H}$

$$U(|\phi\rangle_A|e\rangle_B) = e^{i\alpha(\phi,e)}|\phi\rangle_A|\phi\rangle_B$$

for some real number α depending on ϕ and e.

One-qubit unitary gates

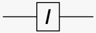
Matrix representation in a basis

Operation from $|i\rangle$ to $|j\rangle$

$$U_{ji} = |j\rangle\langle i|$$

Identity gate

Before	After
$|0\rangle$	$|0\rangle$
$|1\rangle$	$|1\rangle$



$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X = |0\rangle\langle 0| + |1\rangle\langle 1| =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

One-qubit unitary gates

Matrix representation in a basis

Operation from $|i\rangle$ to $|j\rangle$

$$U_{ji} = |j\rangle\langle i|$$

NOT gate

Before	After
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X = |1\rangle\langle 0| + |0\rangle\langle 1| =$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

One-qubit unitary gates

Hadamard gate

Before	After
$\lvert 0\rangle$	$\frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)$
$\lvert 1\rangle$	$\frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)$

$$H = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 1| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Hermitian and Unitary

$$H \cdot H^\dagger = H \cdot H = \mathbb{I}$$

- Changes basis

$$HXH = Z$$

- Hadamard transform

$$H^{\otimes n}(|0\rangle)^{\otimes n} = H \otimes \cdots \otimes H|0\ldots 0\rangle = \frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1}|i\rangle$$

One-qubit unitary gates

Matrix representation in a basis

Operation from $|i\rangle$ to $|j\rangle$

$$U_{ji} = |j\rangle\langle i|$$

Phase gate - Exercise 4

Before	After
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$i\|1\rangle$

What is the matrix representation of S?

$$a) \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, b) \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, c) \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, d) \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$$

One-qubit unitary gates

Matrix representation in a basis

Operation from $|i\rangle$ to $|j\rangle$

$$U_{ji} = |j\rangle\langle i|$$

Phase gate - Exercise

Before	After
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$i\|1\rangle$

What is the matrix representation of S?

$$b) \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

Gates as an exponentiation of gates

We (theoretically) can perform arbitrary continuously parameterized gates

$$R_X(\theta) = e^{-iX\frac{\theta}{2}}$$

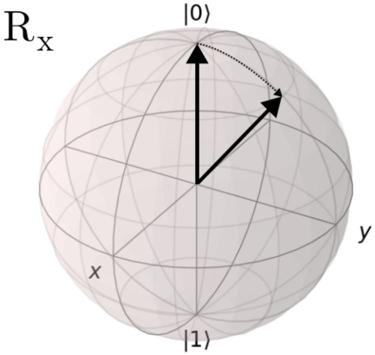
$$R_X(\theta) = \begin{bmatrix} cos(\theta/2) & -i\ sin(\theta/2) \\ -i\ sin(\theta/2) & cos(\theta/2) \end{bmatrix}$$

$$R_Y(\theta) = e^{-iY\frac{\theta}{2}}$$

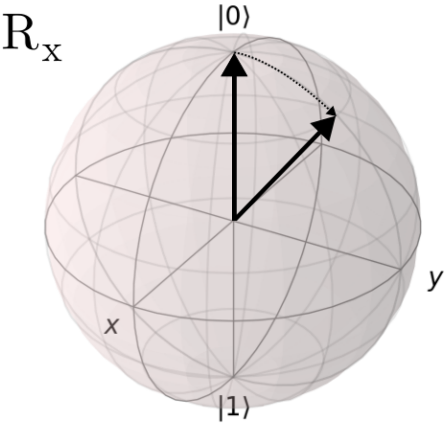
$$R_Y(\theta) = \begin{bmatrix} cos(\theta/2) & -sin(\theta/2) \\ sin(\theta/2) & cos(\theta/2) \end{bmatrix}$$

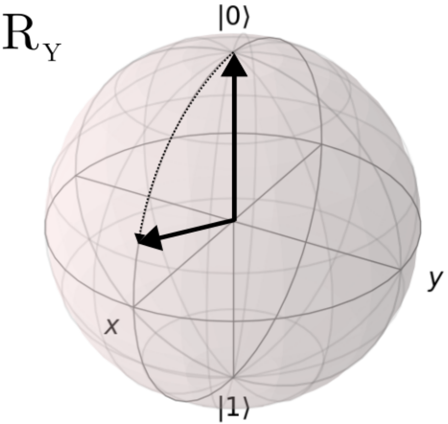
$$R_Z(\theta) = e^{-iZ\frac{\theta}{2}}$$

$$R_Z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$



Quantum computing as unitary rotations

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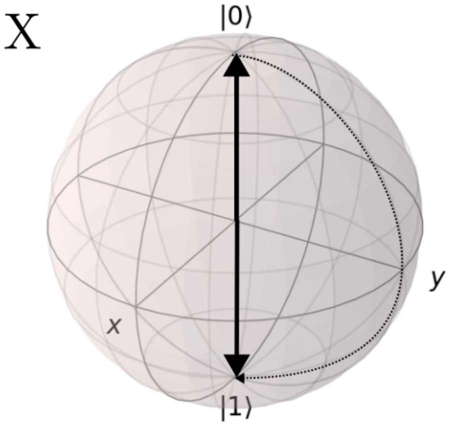
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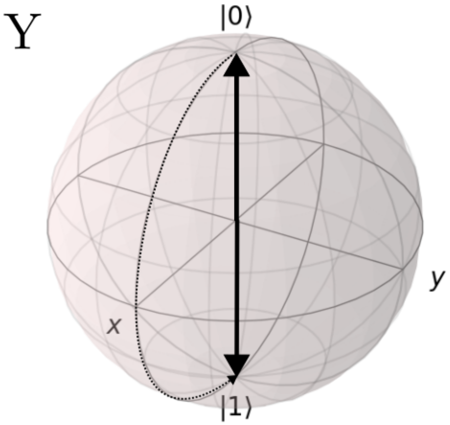
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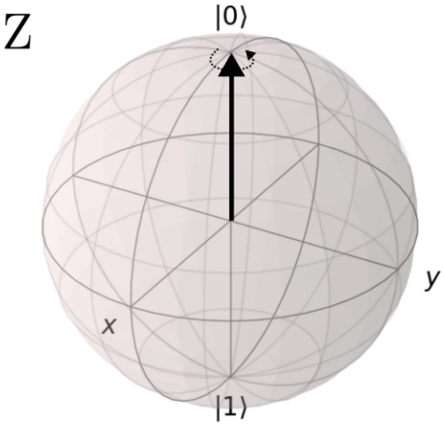
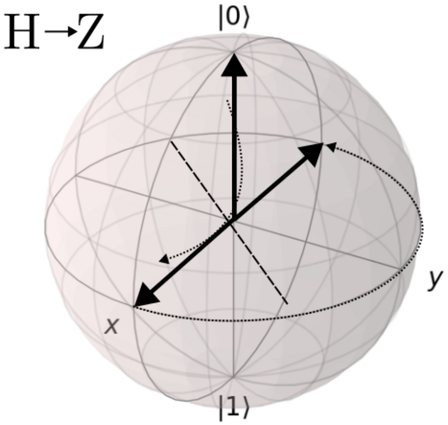
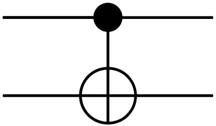
Quantum computing as unitary rotations




H→Z
|0⟩
|1⟩
x
y

Controlled NOT gate



CNOT truth table

Before		After	
C	T	C	T
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

Matrix representation

$$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, |01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, |11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

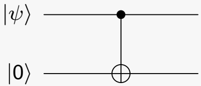
$$CNOT = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11| =$$

$$= \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix}$$

CNOT gate - action on an arbitrary quantum state

Arbitrary state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



$$CNOT(\alpha|0\rangle + \beta|1\rangle) \otimes (|0\rangle) =$$

$$= CNOT(\alpha|00\rangle + \beta|10\rangle) =$$

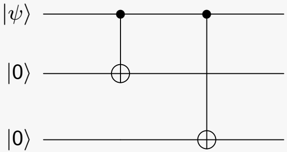
$$= \alpha|00\rangle + \beta|11\rangle$$

For $\alpha = \frac{1}{\sqrt{2}}, \beta = \frac{1}{\sqrt{2}}$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Even more qubits

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle$$

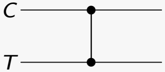


$$\alpha|000\rangle + \beta|111\rangle$$

Error correction - repetition codes

Controlled Z gate

diagram



CNOT truth table

Before		After	
C	T	C	T
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$-\|1\rangle$

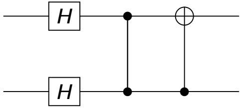
Matrix representation

$$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, |01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, |11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

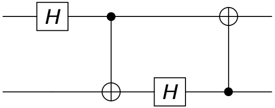
$$CZ = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 11| - |11\rangle\langle 11| =$$

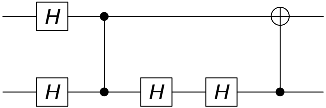
$$= \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & -1\end{bmatrix}$$

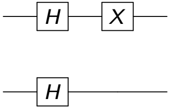
Exercise 4 Quantum circuit equivalence



Which of the following quantum circuits are equivalent to the one above?

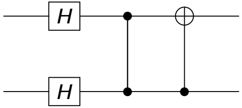
1)


2)


3)


Answers: a) All, b) Only 2, c)1 and 2, d) Only 3

Exercise 4 Quantum circuit equivalence



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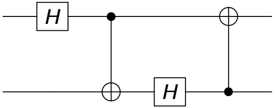
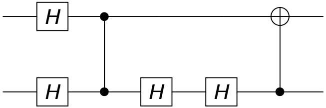
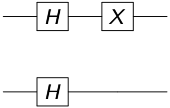


2)



3)



Answers: a) All, b) Only 2, c)1 and 2, d) Only 3

$$HH = \mathbb{I}, HXH = Z$$